# AB-3151

## Third Year B. Sc. (Sem. V) Examination

**March / April – 2015**

## Mathematics : Paper - MTH-501

*(Group Theory)*

Time : 3 Hours] [Total Marks : **70**

**Instructions :**

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
**Fillup strictly the details of ☛ signs on your answer book.**

Name of the Examination :
☛ **T. Y. B. Sc. (SEM. 5)**

Name of the Subject :
☛ **MATHEMATICS - MTH-501**

☛ Subject Code No. : **3 1 5 1** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) All questions are compulsory.

(3) Figures to the right indicate the marks of the question.

(4) Follow usual notations.

1 Answer the following : (any **five**) **10**

(1) Justify : If $a|b+c$, then $a|b$ and $a|c$.

(2) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then prove that $ac \equiv bd \pmod{n}$.

(3) Prove that the identity element in a group $G$ is unique.

(4) If $G$ is an abelian group, then prove that, $(ab)^2 = a^2b^2$, for all $a, b \in G$.

(5) If $G$ is a finite group and $a \in G$, then prove that $a^{0(G)} = e$.

(6) Find the remainder when $2^{29}$ is divided by 29.

(7) If for a subgroup $N$ of a group $G$; $Na.Nb = Nab,\ \forall a, b \in G$, then prove that $N$ is a normal subgroup of $G$.

(8) Find the Kernel of a homomorphism $\phi : G \to \overline{G}$ defined by $\phi(x) = 2^x, \forall\, x \in G;$ where $G$ is the group of all real numbers under addition and $\overline{G}$ is the group of all positive real numbers under multiplication.

2 (a) Define relatively prime integers. If $a|x$ and $b|x$ with $a$ and $b$ are relatively prime integers, then prove that $ab|x$. **5**

**OR**

(a) Prove that the congruence relation modulo $n$ has $n$ distinct congruence classes modulo $n$. **5**

(b) Attempt any **two** : **10**

(1) State and prove Euclid's Lemma.

(2) Find integers $m$, $n$ satisfying $(1128, 33) = 1128\,m + 33\,n$. Also, find $[1128, 33]$.

(3) Prove that :

(i) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a - c \equiv b - d \pmod{n}$.

(ii) If $a.b \equiv a.c \pmod{n}$ and $a$ and $n$ are relatively prime then $b \equiv c \pmod{n}$

(4) Prove that $n$ is a prime number in $Jn$ if and only if whenever $[a][b] = [0]$, then either $[a] = [0]$ or $[b] = [0]$.

3 (a) In a group $G$, prove that : **5**

(i) $\left(a^{-1}\right)^{-1} = a, \forall\, a \in G;$

(ii) $(a.b)^{-1} = b^{-1}.a^{-1}, \forall\, a, b \in G$.

**OR**

(a) Let $H$ be a non-empty subset of a group $G$. **5**
Then prove that $H$ is a subgroup of $G$ if and only if

(i) $a, b \in H$ implies that $a.b \in H$;

(ii) $a \in H$ implies that $a^{-1} \in H$.

(b) Attempt any **two** : **10**

(1) Prove that $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| \begin{matrix} a^2 + b^2 \neq 0 \\ a, b \in R \end{matrix} \right\}$ is a group under matrix multiplication.

(2) Prove that a group of order 4 (four) is abelian.

(3) If $H$ is a finite subset of a group $G$, closed under the multiplication in $G$, then prove that $H$ is a subgroup $G$.

(4) Prove that the centre of a group $G$ is a subgroup of $G$.

4 (a) Let $H$ be a subgroup of a group $G$. Then prove the relation $a \equiv b \bmod H$; where $a, b \in G$; is an equivalence relation on $G$. **5**

**OR**

(a) Let $H$ and $K$ be subgroups of a group $G$. If $HK$ is a subgroup of $G$, then prove that $HK = KH$. State the formula for O($HK$) **5**

(b) Attempt any two : **10**

(1) If $H$ is a subgroup of a finite group $G$, then prove that $O(H) | O(G)$.

(2) Define a cyclic group. Prove that the group $U_8$ is not cyclic.

(3) Prove that every group of prime order is cyclic.

(4) If in a group $G$, $a^5 = e$ and $aba^{-1} = b^2$; for $a, b \in G$; then find $O(b)$.

5 (a) Prove that a subgroup $N$ of a group $G$ is a normal subgroup of $G$ if and only if $gNg^{-1} = N, \ \forall \ g \in G$. **5**

**OR**

(a) Let $\phi : G \to \overline{G}$ be a homomorphism of a group $G$ in to a group $\overline{G}$. Then prove that : **5**

(i) $\phi(e) = \overline{e}$;

(ii) $\phi(x^{-1}) = [\phi(x)]^{-1}, \forall x \in G$.

(b) Attempt any **two** : **10**

(1) Define a normal subgroup. Prove that every subgroup of an abelian group is normal.

(2) If $N$ and $M$ are subgroups of a group $G$ and $N \cap M = (e)$, then prove that $nm = mn, \ \forall \ n \in N, \ \forall \ m \in M$.

(3) Let $\phi$ be a homomorphism of a group $G$ onto a group $\overline{G}$ with kernel $K$. Then prove that $G|K$ is isomorphic to $\overline{G}$.

(4) Prove that a homomorphism $\phi$ of a group $G$ into a group $\overline{G}$ with kernel $K_\phi$ is an isomorphism if and only if $K_\phi = (e)$.