AB-3151
Third Year B. Sc. (Sem. V) Examination
March / April – 2015
Mathematics : Paper - MTH-501
(Group Theory)

Time : 3 Hours] [Total Marks : 70

Instructions :
(1) Fill up strictly the details of the sign on your answer book.

Name of the Examination : T. Y. B. Sc. (SEM. 5)
Name of the Subject : MATHEMATICS - MTH-501
Subject Code No. : 3 1 5 1

Seat No. :

(2) All questions are compulsory.
(3) Figures to the right indicate the marks of the question.
(4) Follow usual notations.

1 Answer the following : (any five) 10

(1) Justify : If $a|b+c$, then $a|b$ and $a|c$.

(2) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then prove that $ac \equiv bd \pmod{n}$.

(3) Prove that the identity element in a group $G$ is unique.

(4) If $G$ is an abelian group, then prove that, $(ab)^2 = a^2b^2$, for all $a, b \in G$.

(5) If $G$ is a finite group and $a \in G$, then prove that $a^{o(G)} = e$.

(6) Find the remainder when $2^{29}$ is divided by 29.

(7) If for a subgroup $N$ of a group $G$; $NaNb = Nab$, $\forall a, b \in G$, then prove that $N$ is a normal subgroup of $G$.

AB-3151] 1 [ Contd.....
(8) Find the Kernel of a homomorphism $\phi : G \to \overline{G}$ defined by $\phi(x) = 2^x, \forall x \in G$; where $G$ is the group of all real numbers under addition and $\overline{G}$ is the group of all positive real numbers under multiplication.

2  (a) Define relatively prime integers. If $a | x$ and $b | x$ with $a$ and $b$ are relatively prime integers, then prove that $ab | x$.

OR

(a) Prove that the congruence relation modulo $n$ has $n$ distinct congruence classes modulo $n$.

(b) Attempt any two:

   (1) State and prove Euclid's Lemma.

   (2) Find integers $m$, $n$ satisfying $(1128, 33) = 1128 m + 33 n$. Also, find $\lfloor 1128, 33 \rfloor$.

   (3) Prove that:

      (i) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a - c \equiv b - d \pmod{n}$.

      (ii) If $a b \equiv a c \pmod{n}$ and $a$ and $n$ are relatively prime then $b \equiv c \pmod{n}$

   (4) Prove that $n$ is a prime number in $\mathbb{Z}$ if and only if whenever $[a][b] = [0]$, then either $[a] = [0]$ or $[b] = [0]$.

3  (a) In a group $G$, prove that:

   (i) $\left(a^{-1}\right)^{-1} = a, \forall a \in G$;

   (ii) $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}, \forall a, b \in G$.

OR

AB-3151] 2 [ Contd.....
(a) Let $H$ be a non-empty subset of a group $G$. Then prove that $H$ is a subgroup of $G$ if and only if
   (i) $a, b \in H$ implies that $ab \in H$;
   (ii) $a \in H$ implies that $a^{-1} \in H$.

(b) Attempt any two:

   (1) Prove that $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \left| \begin{array}{c} a^2 + b^2 \neq 0 \\ a, b \in \mathbb{R} \end{array} \right. \right\}$ is a group under matrix multiplication.

   (2) Prove that a group of order 4 (four) is abelian.

   (3) If $H$ is a finite subset of a group $G$, closed under the multiplication in $G$, then prove that $H$ is a subgroup $G$.

   (4) Prove that the centre of a group $G$ is a subgroup of $G$.

(a) Let $H$ be a subgroup of a group $G$. Then prove the relation $a \equiv b \mod H$, where $a, b \in G$, is an equivalence relation on $G$.

OR

(a) Let $H$ and $K$ be subgroups of a group $G$. If $HK$ is a subgroup of $G$, then prove that $HK = KH$. State the formula for $O(HK)$

(b) Attempt any two:

   (1) If $H$ is a subgroup of a finite group $G$, then prove that $O(H) | O(G)$.

   (2) Define a cyclic group. Prove that the group $U_8$ is not cyclic.

   (3) Prove that every group of prime order is cyclic.

   (4) If in a group $G$, $a^5 = e$ and $aba^{-1} = b^2$; for $a, b \in G$; then find $O(b)$.
5 (a) Prove that a subgroup \( N \) of a group \( G \) is a normal subgroup of \( G \) if and only if \( gNg^{-1} = N, \forall g \in G \).

OR

(a) Let \( \phi : G \rightarrow \overline{G} \) be a homomorphism of a group \( G \) into a group \( \overline{G} \). Then prove that:

(i) \( \phi(e) = \overline{e} \);

(ii) \( \phi(x^{-1}) = [\phi(x)]^{-1}, \forall x \in G \).

(b) Attempt any two:

(1) Define a normal subgroup. Prove that every subgroup of an abelian group is normal.

(2) If \( N \) and \( M \) are subgroups of a group \( G \) and \( N \cap M = \{e\} \), then prove that \( nm = mn, \forall n \in N, \forall m \in M \).

(3) Let \( \phi \) be a homomorphism of a group \( G \) onto a group \( \overline{G} \) with kernel \( K \). Then prove that \( G/K \) is isomorphic to \( \overline{G} \).

(4) Prove that a homomorphism \( \phi \) of a group \( G \) into a group \( \overline{G} \) with kernel \( K_\phi \) is an isomorphism if and only if \( K_\phi = \{e\} \).