AB-3152
Third Year B. Sc. (Sem. V) Examination
March / April – 2015
Mathematics : Paper - MTH-502
(Linear Algebra - I)

Time : 3 Hours] [Total Marks : 70

Instructions :
(1)

(2) All questions are compulsory.
(3) Figures to the right indicate the marks of the question.
(4) Follow usual notations.

1 Answer the following : (any five) 10

(1) In a vector space \( V \), prove that \((-1)u = -u\), for every vector \( u \) in \( V \).

(2) In a vector space \( V \), prove that \(-u - v = -v - u\), for all vectors \( u, v \) in \( V \).

(3) Give an example of two distinct subspaces \( U \) and \( W \) of a vector space \( V \) such that \( U \cap W \) is a subspace of \( V \).

(4) Find the span of the set \( S = \{(1,1,0),(0,0,-1)\} \) of vectors in \( V_3 \). Does \((1,1,1)\) belong to \([S]\) ?

(5) Are the vectors \((0,-1,1)\) and \((1,0,1)\) collinear in \( V_3 \) ? Justify your answer.

(6) Is the set \( \{(1,1,0),(0,1,1),(-1,0,1)\} \) linearly independent in \( V_3 \) ?

[ Contd.....]
(7) Find the co-ordinate vector of a vector \((2, 3, 4, -1)\)
relative to the ordered basis
\[B = \{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 0)\}\]
for \(V_4\).

(8) Which is the basis for the vector space \(\{0\}\)? Find the
dimension of the vector space \(\{0\}\).

2 (a) Let \(\mathbb{R}^+\) be the set of all positive real numbers. The
operations of addition and scalar multiplication in \(\mathbb{R}^+\)
are defined as follows:
\[u + v = uv\]
and
\[\alpha \cdot u = u^\alpha; \text{ for all } u, v \text{ in } \mathbb{R}^+ \text{ and for every } \alpha \in \mathbb{R}.\]

OR

(a) If a non-empty subset \(S\) of a vector space \(V\) satisfies
the conditions:
(i) \(u, v \in S \implies u + v \in S\);
(ii) \(u \in S\) and \(\alpha\) any scalar \(\implies \alpha \cdot u \in S\), then prove that
\(S\) is a subspace of \(V\).

(b) Attempt any two:

(1) In a vector space \(V\), prove that \(\alpha \cdot u = 0\) if and only
if either \(\alpha = 0\) or \(u = 0\).

(2) Give the reason why the following subsets of \(V_3\)
are not subspaces of \(V_3\):

(i) \[S = \{(x, y, z) \in V_3 \mid z \text{ is an integer}\};\]

(ii) \[S = \{(x, y, z) \in V_3 \mid x + y + z \geq 0\}.\]

(3) If \(u_0\) is a fixed vector in a vector space \(V\), then
prove that
\[[u_0] = \{\alpha \cdot u_0 \mid \alpha \text{ any Scalar}\}\]
is a subspace of \(V\).

(4) Prove that the set \(S = \{(x_1, x_2, x_3) \in V_3 \mid x_1 = 0\}\)
is a
subspace of \(V_3\).
3 (a) If \( S \) is a non-empty subset of a vector space \( V \), then prove that \([S]\) is a subspace of \( V \).

OR

(a) If \( U \) and \( W \) are two subspaces of a vector space \( V \), then prove that \( U + W = [U \cap W] \).

(b) Attempt any two:

(1) If \( v_1, v_2, \ldots, v_n \) are \( n \) vectors of a vector space \( V \), then prove that:

(i) \[
\begin{bmatrix}
v_1, v_2, \ldots, v_n
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \cdot v_1, \alpha_2 \cdot v_2, \ldots, \alpha_n \cdot v_n
\end{bmatrix},
\]
where each \( \alpha_i \neq 0 \);

(ii) \[
\begin{bmatrix}
v_1 - v_2, v_1 + v_2
\end{bmatrix} = \begin{bmatrix}
v_1, v_2
\end{bmatrix}.
\]

(2) If \( S \) is a non-empty subset of a vector space \( V \), then prove that the subspace \([S]\) is the smallest subspace of \( V \) containing \( S \).

(3) If \( U \) and \( W \) are two subspaces of a vector space \( V \), then prove that \( U \cap W \) is a subspace of \( V \).

(4) Find the span of the set
\[
S = \{(1, 2, 1), (1, 1, -1), (4, 3, -2)\}
\]
of vectors in \( V_3 \). Does \((1, 0, 1)\) belong to \([S]\) ?

4 (a) Define an LD set. Prove that any set of vectors in a vector space \( V \) containing the zero vector is LD.

OR

(a) In a vector space \( V \), suppose \( \{v_1, v_2, \ldots, v_n\} \) is an ordered set of vectors with \( v_1 \neq \emptyset \). If the set \( \{v_1, v_2, \ldots, v_n\} \) is LD, then prove that
\[
v_k \in [v_1, v_2, \ldots, v_{k-1}], \quad 2 \leq k \leq n
\]

(b) Attempt any two:

(1) In a vector space \( V \), if \( v \in [v_1, v_2, \ldots, v_n] \), then prove that the set \( \{v, v_1, v_2, \ldots, v_n\} \) is LD.
(2) Check whether the set
\[ \{(1, 0, 0, 0), (1, 1, 0, 0), (0, 0, 1, 0), (1, 1, 1, 1)\} \] of vectors in
\( V_4 \) is LI or LD. If LD, then express one of the
vectors which is a linear combination of its
preceding vectors.

(3) Prove that the set \( \{v_1, v_2, v_3\} \) is LD in a vector
space \( V \) if and only if \( v_1, v_2, v_3 \) are coplanar.

(4) Define an LI set. If \( u, v, w \) are LI vectors of a vector
space \( V \), then what do you say about the vectors
\( u - v, \ v - w \) and \( w - u \)? Justify your answer.

5 (a) Define a basis for a vector space. Prove that if a
vector space \( V \) has a basis of \( n \) elements, then every
other basis for \( V \) also has \( n \) elements.

OR

(a) Prove that in an \( n \)-dimensional vector space \( V \), any
set of \( n \) LI vectors is a basis for \( V \).

(b) Attempt any two:

(1) Define the dimension of a vector space. If \( V \) is an
\( n \)-dimensional vector space, then prove that every
set of \( p \) vectors; with \( p > n \), is LD.

(2) In a vector space \( V \), the set \( \{v_1, v_2, \ldots, v_n\} \) generates
\( V \). If \( \{v_1, v_2, \ldots, v_n\} \) is LI, then prove that the
expression for any \( v \) in \( V \); namely;
\[ v = \alpha_1 \cdot v_1 + \alpha_2 \cdot v_2 + \ldots + \alpha_n \cdot v_n, \]
for some scalars
\( \alpha_1, \alpha_2, \ldots, \alpha_n \); is unique.

(3) Prove that the set
\[ B = \{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 0)\} \]

of vectors in \( V_4 \) is a basis for \( V_4 \).

(4) Extend an LI set \( \{(1, 1, 1), (1, 2, 1, 2)\} \) of vectors in
\( V_4 \) to the basis for \( V_4 \).