AD-3249
Third Year B. Sc. (Sem. VI) Examination
March / April – 2014
Mathematics : MTH-601
(Ring Theory)

Time : 3 Hours] [Total Marks : 70

Instructions :

(1)

(2) All questions are compulsory.

(3) Figures to the right indicate marks of the questions.

(4) Follow usual notations.

1 Answer the following as directed of any Five : 10

(1) If R is a ring, then for all \( a, b \in R \), prove that
\[
(-a) \cdot b = -a \cdot b.
\]

(2) In a Boolean ring \( R \); prove that \( a + a = 0 \); for every
\( a \in R \).

(3) Find all ideals of the ring \( J_6 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5} \} \).

(4) If \( U \) is an ideal of a ring \( R \) with unit element 1 and
\( 1 \in U \), then prove that \( U = R \).

(5) Give an example of a Euclidean ring, which is not a field.

AD-3249] 1

[Contd...]
(6) Define "Unit" in a commutative ring with a unit element 1. Mention all the units of the ring \( \mathbb{Z} \) of all integers.

(7) Find all associates of 5 in the ring \( J_8 \).

(8) Define "a Prime Element" and "Relatively Prime Elements" in a Euclidean ring.

2 (a) Prove that every finite integral domain is a field.  

OR

(a) Define "an Integral Domain". Find all the zero divisors in the ring \( J_8 \); of integers modulo 8. Is \( J_8 \) an integral domain?

(b) Attempt any two:

(1) Define "a field". Prove that every field is an integral domain.

(2) If \( p \) is a prime number, then prove that \( J_p \); the ring of integers modulo \( p \); is a field.

(3) Prove that a Boolean ring is commutative.

(4) Prove that the commutative ring \( D \) is an integral domain if and only if for \( a, b, c \in D \) with \( a \neq 0, a \cdot b = a \cdot c \Rightarrow b = c \).

3 (a) Let \( \phi : R \rightarrow R' \) be a homomorphism of a ring \( R \) into a ring \( R' \). Then prove that:

(i) \( \phi(0) = 0 \);

(ii) \( \phi(\neg a) = \neg \phi(a) \);

for every \( a \in R \).

OR

(a) Let \( R \) be a commutative ring with a unit element 1; whose only ideals are (0) and \( R \) itself. Prove that \( R \) is a field.
(b) Attempt any two:

1. If $\phi : R \to R'$ is a homomorphism with kernel $I(\phi)$ then prove that $I(\phi)$ is an ideal of $R$.

2. Prove that a homomorphism $\phi : R \to R'$ is an isomorphism if and only if $I(\phi) = (0)$.

3. If $F$ is a field, then prove that its only ideals are $(0)$ and $F$ itself.

4. If $R$ is a ring and $a$ is a fixed element of $R$, then prove that

$$aR = \{a \cdot r | r \in R\}$$

is a right ideal of $R$.

4 (a) Prove that every field is a Euclidean ring.

OR

(a) Let $R$ be a Euclidean ring. Prove that any two elements $a$ and $b$ in $R$ have a greatest common divisor $d$. Moreover;

$$d = \lambda \cdot a + \mu \cdot b; \text{ for some } \lambda, \mu \in R.$$

(b) Attempt any two:

1. Let $R$ be a Euclidean ring and let $A$ be an ideal of $R$. Prove that there exists an element $a_0 \in A$ such that $A = (a_0)$.

2. Let $R$ be an integral domain with a unit element and suppose that for $a, b \in R$ both $a | b$ and $b | a$. Prove that $a$ and $b$ are associates.

3. In a commutative ring $R$ with a unit element 1; prove that the relation "$a$ is an associate of $b$" is an equivalence relation.

4. In a Euclidean ring; prove that any two greatest common divisors of given two elements are associates.
(a) Let $R$ be a Euclidean ring and $a, b \in R$. If $b \neq 0$ is a unit in $R$, then prove that $d(a) = d(a \cdot b)$.

OR

(a) Let $A = (a_0)$ be an ideal of a Euclidean ring $R$. If $A = (a_0)$ is a maximal ideal of $R$, then prove that $a_0$ is a prime element in $R$.

(b) Attempt any two:

1. Let $R$ be a Euclidean ring. Prove that every element in $R$ is either a unit in $R$ or can be written as a product of a finite number of prime elements in $R$.

2. Prove that the necessary and sufficient condition that an element $a$ in a Euclidean ring $R$ is unit is that $d(a) = d(1)$.

3. Let $R$ be a Euclidean ring. Suppose for $a, b, c \in R$, $a \mid b \cdot c$ but $(a, b) = 1$. Prove that $a \mid c$.

4. If $\pi$ is a prime element in a Euclidean ring $R$, then prove that:

   (i) For any $a$ in $R$ either $(\pi, a) = 1$ or $\pi \mid a$.

   (ii) $\pi \mid a \cdot b \Rightarrow$ either $\pi \mid a$ or $\pi \mid b$. 
