AD-3249

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Third Year B. Sc. (Sem. VI) Examination

March / April – 2014

Mathematics : MTH-601

(Ring Theory)

Time : **3** Hours] [Total Marks : **70**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fillup strictly the details of ☛ signs on your answer book.	Seat No. :
Name of the Examination : ☛ THIRD YEAR B. SC. (SEM. VI)	
Name of the Subject : ☛ MATHEMATICS : MTH-601 (RING THEORY)	
☛ Subject Code No. : 3 2 4 9 ☛ Section No. (1, 2,.....) : Nil	Student's Signature

(2) All questions are compulsory.

(3) Figures to the right indicate marks of the questions.

(4) Follow usual notations.

1 Answer the following as directed of any **Five** : **10**

(1) If R is a ring, then for all $a, b \in R$, prove that $(-a) \cdot b = -a \cdot b$.

(2) In a Boolean ring R; prove that $a + a = 0$; for every $a \in R$.

(3) Find all ideals of the ring $J_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$.

(4) If U is an ideal of a ring R with unit element 1 and $1 \in U$, then prove that $U = R$.

(5) Give an example of a Euclidean ring, which is not a field.

(6) Define "Unit" in a commutative ring with a unit element 1. Mention all the units of the ring Z of all integers.

(7) Find all associates of $\bar{5}$ in the ring J_8.

(8) Define "a Prime Element" and "Relatively Prime Elements" in a Euclidean ring.

2 (a) Prove that every finite integral domain is a field. **5**

OR

(a) Define "an Integral Domain". Find all the zero divisors in the ring J_8; of integers modulo 8. Is J_8 an integral domain ? **5**

(b) Attempt any **two** : **10**

(1) Define "a field". Prove that every field is an integral domain.

(2) If p is a prime number, then prove that J_p; the ring of integers modulo p; is a field.

(3) Prove that a Boolean ring is commutative.

(4) Prove that the commutative ring D is an integral domain if and only if for a, b, $c \in D$ with $a \neq 0$, $a \cdot b = a \cdot c \Rightarrow b = c$.

3 (a) Let $\phi : R \to R'$ be a homomorphism of a ring R into a ring R'. Then prove that : **5**

(i) $\phi(0) = 0$;

(ii) $\phi(-a) = -\phi(a)$;

for every $a \in R$.

OR

(a) Let R be a commutative ring with a unit element 1; whose only ideals are (0) and R itself. Prove that R is a field. **5**

(b) Attempt any **two** : **10**

(1) If $\phi : R \rightarrow R'$ is a homomorphism with kernel $I(\phi)$ then prove that $I(\phi)$ is an ideal of R.

(2) Prove that a homomorphism $\phi : R \rightarrow R'$ is an isomorphism if and only if $I(\phi) = (o)$.

(3) If F is a field, then prove that its only ideals are (o) and F itself.

(4) If R is a ring and a is a fixed element of R, then prove that

$aR = \{a \cdot r / r \in R\}$ is a right ideal of R.

4 (a) Prove that every field is a Euclidean ring. **5**

OR

(a) Let R be a Euclidean ring. Prove that any two elements a and b in R have a greatest common divisor d. Moreover; **5**

$d = \lambda \cdot a + \mu \cdot b$; for some $\lambda, \mu \in R$.

(b) Attempt any **two** : **10**

(1) Let R be a Euclidean ring and let A be an ideal of R. Prove that there exists an element $a_0 \in A$ such that $A = (a_0)$.

(2) Let R be an integral domain with a unit element and suppose that for $a, b \in R$ both $a | b$ and $b | a$. Prove that a and b are associates.

(3) In a commutative ring R with a unit element 1; prove that the relation "a is an associate of b" is an equivalence relation.

(4) In a Euclidean ring; prove that any two greatest common divisors of given two elements are associates.

5 (a) Let R be a Euclidean ring and $a, b \in R$. If $b \neq 0$ is unit in R, then prove that $d(a) = d(a \cdot b)$. 5

OR

(a) Let $A = (a_0)$ bean ideal of a Euclidean ring R. If $A = (a_0)$ is a maximal ideal of R, then prove that a_0 is a prime element in R. 5

(b) Attempt any **two** : 10

(1) Let R be a Euclidean ring. Prove that every element in R is either a unit in R or can be written as a product of a finite number of prime elements in R.

(2) Prove that the necessary and sufficient condition that an element a in a Euclidean ring R is unit is that $d(a) = d(1)$.

(3) Let R be a Euclidean ring. Suppose for $a, b, c \in R$, $a \mid b \cdot c$ but $(a, b) = 1$. Prove that $a \mid c$.

(4) If π is a prime element in a Euclidean ring R, then prove that :

(i) For any a in R either $(\pi, a) = 1$ or $\pi \mid a$.

(ii) $\pi \mid a \cdot b \Rightarrow$ either $\pi \mid a$ or $\pi \mid b$.