DG-3151
Third Year B. Sc. (Sem. V) Examination
March/April – 2016
Mathematics : Paper - MTH-501
(Group Theory)

Time : 2 Hours] [Total Marks : 50

Instructions :
(1) Fill up strictly the details of signs on your answer book.
(2) First question is compulsory.
(3) Figures to the right indicate full marks of the question.
(4) Follow usual notations.
(5) Use of non-programmable scientific calculator is allowed.

Que 1: Answer any FIVE of the following: (10)
1) If \( b \) is a divisor of \( g \) and of \( h \), show that it is a divisor of \( mg + nh \).
2) If \( a \mid b \) and \( b \mid a \) then show that \( b = \pm a \).
3) If \( G \) is a group, then show that the identity element of \( G \) is unique.
4) Show that if every element of the group \( G \) is its own inverse, then \( G \) is abelian.
5) Does the union of two subgroups is also a subgroup? Justify your answer.
6) Justify: If \( o(G) = 12 \) then \( G \) may have a subgroup of order 8.
7) Prove that every subgroup of an Abelian group is normal.
8) If \( G, \bar{G} \) are groups of real numbers under addition and \( \phi : G \to \bar{G} \) is defined by \( \phi(x) = x + 2 \), for all \( x \in G \), then check whether \( \phi \) is homomorphism.
Que 2: Answer any TWO of the following:

(1) If $a, b$ are integers, not both 0, then prove that $(a, b)$ exists. Moreover, show that there exists integers $m_0$ and $n_0$ such that $(a, b) = m_0a + n_0b$.

(2) Prove that the relation "congruence modulo $n$" defines an equivalence relation on the set of integers.

(3) If $p$ is prime and $p \nmid a$, then prove that there exists an element $[b] \in J_p$ such that $[a][b] = [1]$.

(4) To check that $n$ is a prime number, prove that it is sufficient to show that it is not divisible by any prime number $p$, such that $p \leq \sqrt{n}$.

Que 3: Answer any TWO of the following:

(1) If $H$ is any arbitrary non-empty subset of a group $G$ then state and prove the necessary and sufficient condition for $H$ to be a subgroup of $G$.

(2) Let $a, b \in G$. Then show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for $x, y \in G$. In particular show that $a \cdot u = a \cdot w$ implies $u = w$ and $u \cdot a = w \cdot a$ implies $u = w$ hold in $G$.

(3) Let $G = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid ad - bc = 1; a, b, c, d \in R \right\}$. Prove that $G$ is a non-abelian group under matrix multiplication.

(4) If $G$ is an abelian group, then for all $a, b \in G$, prove that $(ab)^n = a^n b^n$, for all positive integers $n$.

Que 4: Answer any TWO of the following:

(1) If $G$ is a finite group and $H$ is a subgroup of $G$, then show that $o(H) \mid o(G)$.

(2) If $o(G) = p$, a prime number then show that $G$ is a cyclic group.

(3) Let $H$ be a subgroup of a group $G$ and $a \in G$. If $aH a^{-1} = \{aha^{-1} \mid h \in H\}$, then show that $aH a^{-1}$ is a subgroup of $G$. Also find $o(aH a^{-1})$, if $H$ is finite.

(4) Prove that $U_{10} = \{1, 3, 7, 9\}$ is a group under the operation `$\times_{10}$'. Is it cyclic? Justify your answer.

Que 5: Answer any TWO of the following:

(1) State and prove fundamental theorem of homomorphism.

(2) Let $N$ be a subgroup of a group $G$. Then show that $N$ is normal in $G$ if and only if product of two right cosets of $N$ in $G$ is again right coset of $N$ in $G$.

(3) If $\phi: G \rightarrow \overline{G}$ is a homomorphism with kernel $K$, then prove that $K$ is a normal subgroup of $G$.

(4) Define: Normal subgroup. Show that the intersection of two normal subgroups of a group $G$ is a normal subgroup of $G$. 