DG-3151

Third Year B. Sc. (Sem. V) Examination

March/April – 2016

Mathematics : Paper - MTH-501

(Group Theory)

Time : 2 Hours] [Total Marks : **50**

Instructions :

(1)

નીચે દર્શાવેલ ☛ નિશાનીવાળી વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fillup strictly the details of ☛ signs on your answer book.

Name of the Examination :
☛ **Third Year B. Sc. (Sem. V)**

Name of the Subject :
☛ **Mathematics : Paper - MTH-501**

☛ Subject Code No. : **3 1 5 1** ☛ Section No. (1, 2,.....) : **Nil**

Seat No. :

Student's Signature

(2) First question is compulsory.

(3) Figures to the right indicate full marks of the question.

(4) Follow usual notations.

(5) Use of non-programmable scientific calculator is allowed.

Que 1: Answer any **FIVE** of the following: **(10)**

1) If b is a divisor of g and of h, show that it is a divisor of $mg + nh$.
2) If $a|b$ and $b|a$ then show that $b = \pm a$.
3) If G is a group, then show that the identity element of G is unique.
4) Show that if every element of the group G is its own inverse, then G is abelian.
5) Does the union of two subgroups is also a subgroup? Justify your answer.
6) Justify: If $o(G) = 12$ then G may have a subgroup of order 8.
7) Prove that every subgroup of an Abelian group is normal.
8) If $G, \bar{G}$ are groups of real numbers under addition and $\phi: G \to \bar{G}$ is defined by $\phi(x) = x + 2$, for all $x \in G$, then check whether ϕ is homomorphism.

Que 2: Answer any **TWO** of the following: **(10)**

(1) If a, b are integers, not both 0, then prove that (a, b) exists. Moreover, show that there exists integers m_0 and n_0 such that $(a, b) = m_0 a + n_0 b$.

(2) Prove that the relation "congruence modulo n" defines an equivalence relation on the set of integers.

(3) If p is prime and $p \nmid a$, then prove that there exists an element $[b] \in J_p$ such that $[a][b] = [1]$.

(4) To check that n is a prime number, prove that it is sufficient to show that it is not divisible by any prime number p, such that $p \leq \sqrt{n}$.

Que 3: Answer any **TWO** of the following: **(10)**

(1) If H is any arbitrary non-empty subset of a group G then state and prove the necessary and sufficient condition for H to be a subgroup of G.

(2) Let $a, b \in G$. Then show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for $x, y \in G$. In particular show that $a \cdot u = a \cdot w$ implies $u = w$ and $u \cdot a = w \cdot a$ implies $u = w$ hold in G.

(3) Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / ad - bc = 1; a, b, c, d \in R \right\}$. Prove that G is a non-abelian group under matrix multiplication.

(4) If G is an abelian group, then for all $a, b \in G$, prove that $(ab)^n = a^n b^n$, for all positive integers n.

Que 4: Answer any **TWO** of the following: **(10)**

(1) If G is a finite group and H is a subgroup of G, then show that $o(H) | o(G)$.

(2) If $o(G) = p$, a prime number then show that G is a cyclic group.

(3) Let H be a subgroup of a group G and $a \in G$. If $aHa^{-1} = \{aha^{-1} \mid h \in H\}$, then show that aHa^{-1} is a subgroup of G. Also find $o(aHa^{-1})$, if H is finite.

(4) Prove that $U_{10} = \{1, 3, 7, 9\}$ is a group under the operation '$\times_{10}$'. Is it cyclic? Justify your answer.

Que 5: Answer any **TWO** of the following: **(10)**

(1) State and prove fundamental theorem of homomorphism.

(2) Let N be a subgroup of a group G. Then show that N is normal in G if and only if product of two right cosets of N in G is again right coset of N in G.

(3) If $\phi: G \to \bar{G}$ is a homomorphism with kernel K, then prove that K is a normal subgroup of G.

(4) Define: Normal subgroup. Show that the intersection of two normal subgroups of a group G is a normal subgroup of G.