1 Answer any FIVE of the following: 10

(1) Define: Zero divisor with an illustration.

(2) Give an example of a ring having 2016 elements. Is it a field? Why?

(3) If \( R \) is a ring, then for all \( a, b \in R \), prove that \( a(-b) = -(ab) \).

(4) Which of the rings \( \mathbb{Z}_{18} \) and \( \mathbb{Z}_{19} \) are integral domains? Why?

(5) If \( U \) is an ideal of a ring \( R \) and \( 1 \in U \) then prove that \( U = R \).

(6) Define: Euclidean Domain.
(7) Let $R$ be a Euclidean ring $R$ and $\pi \in R$ is a prime element. If $\pi | ab$, $a, b \in R$ then show that $\pi | a$ or $\pi | b$.

(8) Find all the associates of $\overline{4}$ in $J_6$.

2 Answer any TWO of the following:

(1) Prove that any finite integral domain is a field.

(2) If $p$ is a prime number, then prove that $J_p$, the ring of integers modulo $p$ is a field.

(3) Let $R$ be the set of integers mod 7 under the addition and multiplication mod 7 defined as follows:

$\overline{i} + \overline{j} = \overline{k}$; where $k$ is the reminder of $i + j$ on division by 7, and $\overline{i} \cdot \overline{j} = \overline{m}$; where $m$ is the reminder of $ij$ on division by 7.

Prove that $R$ is a commutative ring.

(4) Prove that any field is an integral domain.

3 Answer any TWO of the following:

(1) Let $R$ be a commutative ring with unit element whose only ideals are (0) and $R$ itself. Then prove that $R$ is a field.

(2) If $\phi : R \rightarrow R'$ is a homomorphism with kernel $I(\phi)$ then prove that $I(\phi)$ is an ideal of $R$.

(3) If $R$ is a ring with unit element 1 and $\phi$ is a homomorphism of $R$ onto $R'$, then prove that $\phi(1)$ is the unit element of $R'$.

(4) Let $U$ be an ideal of $R$. Let

$r(U) = \{x \in R / xu = 0, \text{ for all } u \in U\}$. Prove that $r(U)$ is an ideal of $R$. 

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DRR-3249] 2
4  Answer any TWO of the following:

(1) Let R be a Euclidean domain and let A be some ideal of R. Then show that there exists an element \( a_0 \in A \) such that A consists exactly of all \( a_0x \); as \( x \) ranges over R.

(2) Let R be a Euclidean ring. Then prove that any two elements \( a, b \in R \) always have a greatest common divisor \( d \) and \( d = \lambda a + \mu b \); for some \( \lambda, \mu \in R \).

(3) Explain what do you mean by unit in a commutative ring. Prove that a Euclidean ring always possesses a unit element.

(4) If R is an integral domain with unit element and if \( a, b \in R \), then find the relation between \( a \) and \( b \) when both \( a|b \), \( b|a \) are true.

5  Answer any TWO of the following:

(1) Let R be a Euclidean ring and \( a_0 \) is a prime element of R. Prove that \( A=(a_0) \) is a maximal ideal of R.

(2) Let R be a Euclidean ring. Prove that every element in R is either a unit in R or can be written as the product of a finite number of prime elements of R.

(3) In a Euclidean ring prove that any two greatest common divisors of two given elements are associates.

(4) Prove that a necessary and sufficient condition that the element \( a \) in the Euclidean ring be a unit is that \( d(a) = d(1) \).