(1) Check the linearity of the map \( T : V_2 \rightarrow V_1 \) defined by 
\[ T(x, y) = x^2 + y^2. \]

(2) Let \( T : U \rightarrow V \) be a linear map. Then prove that 
\[ T(-u) = -T(u), \] for every \( u \in U. \)

(3) Define : Null space and Nullity of a linear map.

(4) Find \( R(T) \); for the linear map \( T : V_2 \rightarrow V_2 \) defined by 
\[ T(e_1) = (0, 0) \] and 
\[ T(e_2) = (0, -1). \]
(5) Prove that the linear map \( T : V_2 \rightarrow V_2 \) defined by
\[ T(e_1) = e_2 \quad \text{and} \quad T(e_2) = e_1 - e_2 \]
is one-one.

(6) Let \( T : U \rightarrow V \) and \( S : V \rightarrow W \) two linear maps. If \( ST \) is onto, then prove that \( S \) is onto.

(7) In an inner product space \( V \), prove that, \( u \cdot 0 = 0 \), for every \( u \in V \).

(8) Define: Orthogonal Vectors in an inner product space. Give an illustration of orthogonal vectors in the inner product space \( V_3 \).

2 Attempt any Two:

(1) Prove that the map \( T : V_3 \rightarrow V_3 \) defined by
\[ T(x_1, x_2, x_3) = (x_1, x_2, 0) \]
is linear.

(2) Let \( T : U \rightarrow V \) be a linear map. Prove that \( R(T) \) is a subspace of a vector space of \( V \).

(3) Obtain the general rule for a linear map \( T : V_2 \rightarrow V_2 \) such that \( T(0,1) = (3,2) \) and \( T(3,1) = (2,2) \).

(4) Let \( T : U \rightarrow V \) be a linear map. Then prove that \( T \) is one-one if and only if \( N(T) = \{0\} \).

3 Attempt any Two:

(1) Define a non-singular map. Let \( T : U \rightarrow V \) be a linear map. If \( v_1, v_2, \ldots, v_n \) are LI vectors of \( R(T) \) and \( u_1, u_2, \ldots, u_n \) are vectors of \( U \) such that \( T(u_1) = v_1, T(u_2) = v_2, \ldots, T(u_n) = v_n \), then prove that \( u_1, u_2, \ldots, u_n \) are LI.
(2) Let $T : U \rightarrow V$ be a linear map and $U$ be a finite-dimensional vector space. Then prove that 
$$r(T) + n(T) = \dim U.$$ 

(3) Verify the Rank-Nullity Theorem for the linear map 
$T : V_3 \rightarrow V_2$ defined by 
$T(e_1) = (2,1)$, $T(e_2) = (0,1)$ and 
$T(e_3) = (1,1)$.

(4) Prove that the linear map $T : V_3 \rightarrow V_3$ defined by 
$T(e_1) = e_1$, $T(e_2) = e_1 + e_2$ and $T(e_3) = e_1 + e_2 + e_3$ is non-singular and find $T^{-1}$.

4 Attempt any Two :

(1) Let $T : U \rightarrow V$ and $S : V \rightarrow W$ two linear maps. If $S$ and $T$ are non-singular, then prove that $ST$ is non-singular and $(ST)^{-1} = T^{-1}S^{-1}$.

(2) Determine the matrix $(T : B_1, B_2)$ for the linear map 
$T : V_3 \rightarrow V_2$ defined by $T(x,y,z) = (x+y, y+z)$ relative to the bases : $B_1 = \{(1,1,1), (1,0,0), (1,1,0)\}$, $B_2 = \{(1,0), (0,1)\}$.

(3) Determine the linear map associated with 
the matrix 
$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$ 
relative to the bases : $B_1 = \{(1,-1,1), (1,2,0), (0,-1,0)\}$, 
$B_2 = \{(1,1), (2,-1)\}$.
(4) Verify the Rank-Nullity Theorem for the matrix
\[ A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 2 \\ 1 & 0 & 1 \end{bmatrix} \]

5 Attempt any Two:

(1) Define an inner product space. In an inner product space $V$, prove that:

(i) $(u + v) \cdot w = u \cdot w + v \cdot w$;

(ii) $u \cdot (\alpha v) = \alpha (u \cdot v)$; for all $u, v \cdot w \in V$ and for any scalar $\alpha$.

(2) Define the norm of a vector in an inner product space. In an inner product space establish the Triangle Inequality.

(3) Prove that an orthogonal set of non-zero vectors in an inner product space is LI.

(4) Orthogonalize the LI set $B = \{(0, 0, 1), (1, 1, 0), (1, 5, 2)\}$ of $V_3$ by the Gram-Schmidt Process.