Instructions:

(1) Fill up strictly the details of signs on your answer book.

(2) First question is compulsory.

(3) Figures to the right indicate marks of corresponding question.

(4) Follow usual notations.

(5) Use of non-programmable scientific calculator is allowed.

1. Answer any FIVE of the following: 10

   (1) Define closed set $E$ and in usual notations prove that $E \subseteq \overline{E}$.

   (2) Prove that any finite subset of a metric space $M$ is closed.

   (3) Define:

   (i) Bounded set in a metric space $M$

   (ii) Diameter of a set.

   (4) Prove that the interval $[1,6]$ is not a connected subset of $\mathbb{R}$.

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(5) Define:
   (i) Cauchy sequence in a metric space M
   (ii) Complete metric space.

(6) State: "Nested interval theorem".

(7) Define:
   (i) Open covering of a set
   (ii) Heine Borel property.

(8) Prove that the closed subset of a compact metric space is compact.

2 Answer any TWO of the following:

(1) Prove the following:
   (i) Let G be an open subset of the metric space M. Then prove that $G' = M - G$ is closed.
   (ii) Let F be a closed subset of the metric space M. Then prove that $F' = M - F$ is open.

(2) If E is a closed subset of a metric space M, then prove that $\overline{E}$ is closed.

(3) Let $f$ be a continuous real-valued function on the metric space M.
   Let $A$ be the set of all $x \in M$ such that $f(x) \geq 0$. Then prove that A is closed.

(4) If $\mathcal{F}$ is any family of closed subsets of a metric space M, then prove that $\bigcap_{F \in \mathcal{F}} F$ is also closed.
3 Answer any TWO of the following:

1. Prove that the range of a continuous function defined on a connected metric space is also connected.

2. If a subset A of the metric space $\langle M, \rho \rangle$ is totally bounded then prove that A is bounded.

3. If A is a connected subset of the metric space M and if $A \subset B \subset \bar{A}$, then prove that B is connected.

4. If $A_1$ and $A_2$ are connected subsets of a metric space M, and if $A_1 \cap A_2 \neq \emptyset$, then prove that $A_1 \cup A_2$ is also connected.

4 Answer any TWO of the following:

1. If $\langle M, \rho \rangle$ is a complete metric space and A is closed subset of M then prove that $\langle A, \rho \rangle$ is also complete.

2. Prove that $R^2$ is complete.

3. Prove that the interval $(0,1)$ with absolute value metric is not a complete metric space, but it is complete with the metric of $R_d$.

4. Give statement of "Picard's fixed point theorem". If $f(x) = x^2; 0 \leq x \leq \frac{1}{3}$, then prove that T is a contraction on $[0, \frac{1}{3}]$. 

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Answer any TWO of the following:

(1) Prove that the metric space \( \langle M, \rho \rangle \) is compact if and only if whenever \( \mathcal{F} \) is a family of closed subsets of \( M \) with Finite Intersection Property, then \( \bigcap_{F \in \mathcal{F}} F \neq \emptyset \).

(2) Prove that the metric space \( \langle M, \rho \rangle \) is compact if and only if every sequence of points in \( M \) has a subsequence converging to a point in \( M \).

(3) If \( A \) and \( B \) are compact subsets of \( \mathbb{R}^1 \), then prove that \( A \times B \) is a compact subset of \( \mathbb{R}^2 \).

(4) Prove that a connected subset of \( \mathbb{R}_d \) is compact. Also give an illustration of a connected subset of \( \mathbb{R}^1 \) that is not compact.