M. Sc. (Sem. III) (SF) (Mathematics) Examination
March / April - 2016
Paper - 6011 : Special Functions - I

Time : 3 Hours] [Total Marks : 70

Instructions :

(1) Fill up strictly the details of signs on your answer book.

Name of the Examination :
M. SC. (SEM. III) (SF) (MATHEMATICS)

Name of the Subject :
PAPER - 6011 : SPECIAL FUNCTIONS - I

(2) Answer all questions.

(3) Figures to the right indicate full marks of the questions.

(4) Follow usual notations and conventions

1 Answer any TWO of the following

(1) Define uniform convergence of the infinite product. If for a positive constant $M_n$ such that $\sum_{n=1}^{\infty} M_n$ is convergent and $|a_n(k)| < M_n$ for all $z$ then prove that product $\prod_{n=1}^{\infty} (1 + a_n)$ is uniformly convergent.

(II) Define absolute convergence of an infinite product. Show that the product $\prod_{n=1}^{\infty} (1 + a_n)$ with zero factor deleted is absolutely convergent iff $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(III) Define: Infinite Product. Show that $\prod_{n=1}^{\infty} \exp(\frac{1}{n})$ diverges. What can you say about an infinite product $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$?

DF-1489] [Contd...
2 Answer any TWO of the following:

(I) Show that the general solution of the hypergeometric differential equation
\[ z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0 \]
valid for \( |1-z| < 1 \) is given by
\[ w = A F(a, b; a+b-c+1; 1-z) + B (1-z)^{c-b} F(c-b, c-a, 1+c-a-b; 1-z) \]

(II) If \( 2b \) is either zero nor a negative integral and if \( |y| < 1/2 \) and \( |y/(1-y)| < 1 \) then show that
\[ (1-y)^{-a} F\left[\frac{a}{2}, \frac{a}{2} + \frac{1}{2}; \frac{y}{1-y}, b + \frac{1}{2}\right] \]
\[ = F\left[\frac{a}{2}, \frac{a}{2}, 2b, \frac{1}{2}\right] \]

(III) If \( a+b+\frac{1}{2} \) is neither zero nor a negative integer and if \( |x| < 1, |4x(1-x)| < 1 \) Prove that
\[ F\left(a, b; a+b+\frac{1}{2}; 4x(1-x)\right) = F\left(2a, 2b; a+b+\frac{1}{2}; x\right) \]

3 Answer any TWO of the following:

(I) Define contiguous function to \( F(a, b; c; z) \) and derive the relations.
   (i) \( (a-c+1)F = aF(a+1) - (c-1)F(a) \)
   (ii) \( [a+(b-c)z]F = a(1-z)F(a+1) - c^{-1}(c-a)(c-b)zF(c+1) \)

(II) Show that \( w = F(a, b; c; z) \) is a solution of the hypergeometric differential equation
\[ z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0 \]

(III) Derive the integral form of the hypergeometric function \( F(a, b; c; z) \).

And obtain \( F(a, b; c; 1) \).

4 Answer any TWO of the following:

(I) If \( Z > 0 \) then prove that the Euler's product can be expressed in the form of
\[ \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}(1 + \frac{1}{n^2})^{-1}\right) \]

(II) If \( n \) is integral and \( z \) is not a negative integer prove that
\[ \lim_{n \to \infty} \frac{(n-1)!n^z}{(z+n)} = 1 \]

(III) Define Euler constant. Show that it exists and less than unity.

5 Answer any TWO of the following:

(I) If \( \text{Re}(Z) > 0 \), then show that
\[ \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}(1 + \frac{1}{n^2})^{-1}\right) = \int_{0}^{\infty} e^{-t}t^{-1}dt \]

DF-1489] 2 [Contd...
(II) If $0 \leq t < n$, $n$ be a positive integer then show that

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2 e^{-t}}{n}$$

(III) With usual notation show that

$$\lim_{z \to \infty} \frac{n! n^z}{z(z+1)(z+2)\ldots(z+n)}$$