DMM-1591
M. Sc. (Mathematics) (Sem. IV) Examination
April / May – 2016
Advanced Ordinary Differential Equations : Paper - 704

Time : 3 Hours] [Total Marks : 70

Instructions : (1)

(2) Attempt all questions.

(3) Figures to the right indicate marks.

(4) Follow the usual notations and conventions.

1 Attempt any two : 14

(1) Prove that all the solutions of $x'(t) = A(t)x(t)$, where $A(t)$ is $n \times n$ continuous matrix on $[0, \infty]$ and $x$ is an $n$ vector, are stable if they are bounded.

(2) Prove that the system $x' = A(t)x$ is restrictively stable iff $\|\Phi(t)\| \leq M, \|\Phi(t)\| \leq M$ for some positive constant $M$ and $t \geq t_0$, that is iff $\|\Phi(t)\Phi^{-1}(s)\| \leq M$ for some $M$ and for all $t \geq 0, s \geq 0$.

(3) If $y' - A^*(t)y$ is stable and $\int_0^T (A(s))ds \leq d < \infty$ for $t \geq 0$ then it is restrictively stable.
2 Attempt any two:

(1) Prove that a system $x'(t) = A(t)x(t)$ is uniformly stable if it is stable and reducible.

(2) Let there exists a positive constant $M \rightarrow \| \Phi(t) \Phi^{-1}(t) \| \leq M, t_0 \leq s \leq t < \infty$ and let $f$ satisfy the inequality $\| f(t) \| \leq r(t) \| x \|$ where $\gamma(t)$ is non-negative continuous function such that $\int_{t_0}^{\infty} \gamma(t) < \infty$ then $\exists$ is positive constant $L \rightarrow \inf \{ t_1 \geq t_0 \}$ every solution $x(t)$ of $x'(t) = A(t)x(t) + f(t,x)$ for which $\| x(t) \| \leq \frac{C}{L}$ is defined and satisfies $\| x(t) \| \leq L \| x(t_1) \|, \forall t_1 \geq t_0$. In addition to this inequality if $\| \Phi(t) \| \rightarrow 0$ as $t \rightarrow \infty$ then $\| x(t) \| \rightarrow 0$ as $t \rightarrow \infty$.

(3) Show that the scalar differential equation $u' = -u^3$ uniformly asymptotically stable but not exponentially asymptotically stable.

3 Attempt any two:

(1) Let $b(t)$ be the continuously differentiable function on $[0, \infty]$ and if $b(t) \rightarrow 0$ as $t \rightarrow \infty$ and $\int_{0}^{\infty} |b(t)| dt < \infty$, then prove that all the solutions of $u'' + (1+b(t))u = 0$ are bounded over $[0, \infty]$.

(2) State and prove Cauchy Schwartz inequality.

(3) For the equation $u'' + (1+b(t))u = 0$ where $b(t) = \frac{2 \sin t}{t} \left( \frac{\sin 2t}{8t} - 1 \right)$, $t > 0$. Show that $u(t) = \exp \left[ \int_{1}^{t} \frac{\sin^2 s}{s} ds \right]$ sint is its solution but it is not bounded.
Attempt any two:

(1) Prove that if all the solutions of \( u^{n} + a(t)u = 0 \) belong to \( L^{2}[0, \infty) \) and \( b(t) \) is on \([0, \infty)\) then all the solutions of \( u^{11} + (a(t) + b(t))u = 0 \) also belong to \( L^{2}[0, \infty) \).

(2) If \( u \) and \( u^{11} \) belong to \( L^{2}[0, \infty) \) then prove that \( u' \) also belongs to \( L^{2}[0, \infty) \).

(3) Show that the critical point \((0,0)\) of the Vander-Pol's equation \( u^{11} + \varepsilon(u^{2} - 1)u' + u = 0 \) where \( \varepsilon \) is a positive constant is always unstable.

Attempt any two:

(1) Show that the solution \( u(t) = u_{0}\exp(-\alpha(t-t_{0})) \) of \( u' = -\alpha u, \alpha > 0 \) is uniformly asymptotically stable.

(2) Draw the phase portrait and determine the type of the critical point for the system

\[
\begin{bmatrix}
x_1' \\
x_2'
\end{bmatrix} = \begin{bmatrix}
-3 & 1 \\
4 & -2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

(3) Test the stability of the solutions \( u(t) = 0 \) and \( u(t) = 1 \) of the equation \( uu' = -u(1-u) \).