Q.1 Answer any TWO of the following 14

(I) With usual notation prove that
\[ p_n(\cos \alpha) = \sum_{k=0}^{n} \frac{\sin \alpha}{\sin \beta} \binom{n}{k} \binom{\sin (\beta - \alpha)}{\sin \alpha}^{n-k} p_k(\cos \beta) \]

(II) Show that
\[ \int_{-1}^{1} x^n p_n(x) \, dx = \frac{n!}{2^n \left( \frac{1}{2} \right)^{n+1}} \]

(III) Define Legendre polynomials \( p_n(x) \) and show that
\[ p_n(-x) = \frac{2}{2^n (\frac{1}{2})^{n+1}} F_1 \left[ -n, n+1; \frac{1+x}{2} \right] \]

Q.2 Answer any TWO of the following: 14

(I) Prove that
\[ \int_{-1}^{1} (1+x)^{\alpha-1} (1-x)^{\beta-1} p_n(x) \, dx = 2^{\alpha+\beta-1} B(\alpha+\beta) \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} \]

DMM-1602] 1 [Contd....
(II) Prove that \[ p_n(x) = \frac{1}{2} \int_0^\pi \left[ x + (x^2 - 1)^{1/2} \cos \phi \right]^n d\phi \]

(III) Prove that \[ x^n = \frac{n!}{2^n} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(2n - 4k + 1) p_{n-2k}(x)}{k! \left( \frac{3}{2} \right)^{n-k}} \]

Q.3 Answer any TWO of the following:

(I) With usual notation prove that \[ H_{2n}(0) = (-1)^n 2^{2n} \left( \frac{1}{2} \right)_n \quad H'_{2n+1}(0) = (-1)^n 2^{2n+1} \left( \frac{n}{2} \right)_n \]

(II) With usual notation prove that \[ \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y) t^n}{n!} = (1 - 4t^2)^{-1/2} \exp \left[ y^2 - \frac{(y - 2xt)^2}{1 - 4t^2} \right] \]

(III) With usual notation prove that \[ \sum_{n=0}^{\infty} \frac{(c)_n H_n(x) t^n}{n!} = (1 - 2xt)^{-c} \quad _2F_0 \left[ \begin{array}{c} c \ 
 \frac{c}{2}, \frac{c}{2} + \frac{1}{2} \ 
 \frac{-4t^2}{(1-2xt)^2} \end{array} ; - \cdots \right] \]

Q.4 Answer any TWO of the following:

(I) Prove that \[ H_n(x) = \frac{n!}{\pi} \int_0^\pi \exp(2x \cos \theta - \cos 2\theta) \cos(2x \sin \theta - \sin 2\theta - n\theta) \ d\theta \]

(II) Prove that \[ H_n(x) \]

\[ = \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{k! \left( \frac{3}{2} \right)^{n-2k}} \frac{(-1)^k n! (2n-4k+1) P_{n-2k}(x)}{k! \left( \frac{3}{2} \right)^{n-2k}} \]
(III) Define the Hermite polynomials. Derive the Rodrigues formula for $H_n(x)$.

Q.5 Answer any TWO of the following:

(I) Define orthogonality of a simple set of real polynomials $\phi_n(x)$. Show that the set $\phi_n(x)$ is orthogonal iff

$$\int_a^b w(x) x^k \phi_n(x) \, dx = 0, \quad k = 0, 1, \ldots, n-1; \quad \text{for } a < x < b.$$

(II) Let $\{\phi_n(x)\}$ is a simple set of real polynomials orthogonal with respect to $w(x) > 0$ on $a < x < b$. Let $h_n$ be the leading coefficient in $\Phi_n(x)$ so that $\Phi_n(x) = h_n x^n + \pi_{n-1}$ where $\pi_{n-1}$ is a polynomial of degree $n-1$. Let $g_k = (\phi_k, \phi_k)$ then prove that

$$\sum_{k=0}^n g^{-1}_k \phi_k(x) \phi_k(y) = \frac{h_n}{g_nh_{n-1}} \left[ \frac{\phi_{n+1}(y) \phi_n(x) - \phi_{n+1}(x) \phi_n(y)}{y-x} \right]$$

(III) Prove that

$$\sum_{n=0}^\infty \frac{P_n(x)}{n!} t^n = e^{\tfrac{t}{1-x^2}}$$

---

DMM-1602] 3 [ 200 ]