DMM-1603
M. Sc. (Mathematics) (Sem. IV) Examination
April/May – 2016
Paper-7012 : Special Functions-IV

Time : Hours] [Total Marks : 70

Instructions : (1)
(2) Answer all questions.
(3) Each question carries 14 marks.
(4) Follow usual notations and conventions.

1 Attempt any two : 14

(i) Prove that
\[
\frac{1}{(1-t)^c} \text{I}_1 \left[ C; \frac{-xt}{1+\alpha; 1-t} \right] = \sum_{n=0}^{\infty} \binom{c}{n} \frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n} t^n.
\]

(ii) Prove that
\[
(1 + \alpha + n) L_n^{(\alpha)}(x) = (n+1) L_{n+1}^{(\alpha)}(x) + x L_n^{(\alpha+1)}(x)
\]
and
\[
D L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x)
\]

(iii) Prove that
\[
D L_n^{(\alpha)}(x) = -\sum_{k=0}^{n-1} L_k^{(\alpha)}(x)
\]

\[
x^n = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n n! L_k^{(\alpha)}(x)}{(1+\alpha)_k (n-k)!}
\]
(i) Prove that \[
\sum_{k=0}^{n} \frac{L_k^{(\alpha)}(x) L_k^{(\alpha)}(y)}{(1+\alpha)_k} = \frac{(n+1)! \left\{ L_{n+1}^{(\alpha)}(y) L_n^{(\alpha)}(x) - L_n^{(\alpha)}(x) L_{n+1}^{(\alpha)}(y) \right\}}{(1+\alpha)_n (x-y)}
\]

(ii) Prove that \[
L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x) L_{n-k}^{(\alpha)}(y).
\]

(iii) Prove that \[
n L_n^{(\alpha)}(x) = (2n-1+\alpha-x) L_{n-1}^{(\alpha)}(x) - (n-1+\alpha) L_{n-2}^{(\alpha)}(x).
\]

3 Attempt any two:

(i) Prove that
\[
\sum_{n=0}^{\alpha} \frac{P_n^{(\alpha,\beta)}(x) t^n}{(1+\alpha)_n (1+\beta)_n} = {}_0F_1 \left[ \begin{array}{c} \alpha \\ 1+\alpha \end{array} \right] \left[ \begin{array}{c} -t(x+1) \\ \frac{1}{2} \end{array} \right] \right] ^n
\]
Hence deduce that \( P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \)

(ii) Prove that
\[
P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta)_{2n}}{(1+\alpha+\beta)_n n!} \left( \frac{x+1}{2} \right)^n 2F_1 \left[ \begin{array}{c} -n, -\beta-n \\ -\alpha-\beta-2n \end{array} \right] \left[ \begin{array}{c} 1 \\ x+1 \end{array} \right]
\]

(iii) Prove that
\[
\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} \left[ P_n^{(\alpha,\beta)}(x) \right]^2 dx
\]
\[
= \frac{2^{\alpha+\beta+1} (1+\alpha+n) (1+\beta+n)}{n! (1+\alpha+\beta+2n) (1+\alpha+\beta+n)}
\]
4 Attempt any two:
   (i) Prove that
   
   \[(x-1) \left[ (\alpha + \beta + n) D P_n^{(\alpha, \beta)}(x) + (\alpha + n) D P_{n-1}^{(\alpha, \beta)}(x) \right] \]
   \[= (\alpha + \beta + n) \left[ n P_n^{(\alpha, \beta)}(x) - (\alpha + n) P_{n-1}^{(\alpha, \beta)}(x) \right] \]
   
   (ii) Prove that
   
   \[\sum_{n=0}^{\infty} P_n^{(x, \beta)}(x) t^n = \rho^{-1} \left( \frac{2}{1+t+\rho} \right)^{\beta} \left( \frac{2}{1-t-\rho} \right)^{\alpha} \]
   where \( \rho = \left( 1 - 2xt + t^2 \right)^{1/2} \)
   
   (iii) With usual notation prove that
   
   \[\text{F}_4 \left( a; b; c, 1-c+a+b; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)} \right) \]
   \[= 2F_1 \left[ \frac{-x}{c}; \frac{-x}{1-x} \right] 2F_1 \left[ \frac{-y}{1-c+a+b}; \frac{-y}{1-y} \right] \]

5 Attempt any two:
   (i) Define an elliptic function. And prove that
      (a) An elliptic function with no poles in a cell is constant.
      (b) An elliptic function of order less than two is constant.
   (ii) The sum of the residues at the poles in a cell of an elliptic function is zero.
   (iii) State and prove the Rodrigue's formula for \( L_n^{(\alpha)}(x) \).