DPP-1391
M. Sc. (Mathematics) (Sem. II) Examination
March/April – 2016
Paper - 502 : Functional Analysis

Time : Hours] [Total Marks :

Instructions :
(1) Attempt all questions.
(2) Figures to the right indicate marks.
(3) Follow the usual notations and conventions.

Q.1 Attempt Any Two:
(a) Define Euclidean space \( \mathbb{R}^n \) and prove that Euclidean space \( \mathbb{R}^n \) is a metric space.

(b) Show that the space \( L^\infty \) is a complete space.

(c) Prove that \( \sum_{j=1}^{\infty} |\xi_j \eta_j| \leq (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} (\sum_{m=1}^{\infty} |\eta_m|^q)^{1/q} \),

Where \( p > 1, q > 1 \) and \( 1/p + 1/q = 1 \)

Q.2 Attempt Any Two:
(a) Define equivalent norms. Prove that the norms \( \|x\|_\infty \) and \( \|x\|_2 \) defined on \( \mathbb{R}^n \) respectively

\[
\|x\|_\infty = \max_{1 \leq i \leq n}|x_i| \text{ and } \|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}
\]

are equivalent.

(b) Prove that every finite dimensional normed space is complete.

(c) Prove that a compact subspace \( M \) of a metric space \( X \) is closed and bounded.
Q.3 Attempt Any Two:

(a) Let $T$ be a linear operator. Show that

(i) The range $\text{R}(T)$ is a vector space.

(ii) the null space $\text{N}(T)$ is a vector space.

(b) Let $Y$ and $Z$ be subspaces of a normed space $X$ of any dimension. Suppose $Y$ is closed and is a proper subset of $Z$. Show that for every real $\theta \in (0,1)$, there exists $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| \geq \theta$, for every $y \in Y$.

(c) Prove that a linear functional $f : C[a,b] \to \mathbb{R}$ defined by

$$f(x) = \int_{a}^{b} x(t) dt \quad \forall x \in C[a,b], \quad \forall t \in [a,b]$$

is bounded and $\|f\| = b - a$.

Q.4 Attempt Any Two:

(a) Prove that an inner product space is a normed space.

(b) Prove that Unitary space $\mathbb{C}^n$ is a Hilbert space.

(c) In an inner product space $X$, prove that $\mathbf{x} \perp \mathbf{y} \Rightarrow \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Does the converse true? Justify.

Q.5 Attempt Any Two:

(a) Let $Y$ be a closed subspace of a Hilbert space $H$. Show that $Y = Y^\perp$.

(b) Let $X$ be an inner product space, then prove that the corresponding norm satisfied:

(i) $|\langle x, y \rangle| \leq \|x\| \|y\|$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

(c) Orthonormalise an LI set $\{(1,1,1), (-1,0,1), (1,-1,0)\}$ by Gram-Schmidt process.