DPP-1395

M. Sc. (Mathematics) (Sem. II) Examination
March/April – 2016
Paper - 506 : Functions of Complex Variable

Time : 3 Hours] [Total Marks : 70

Instructions :
1. Fill up strictly the details of signs on your answer book.
2. Figures to the right indicate marks.
3. Follow the usual notations and conventions.

Q1
A Define Laurent’s series about z=0. Also if f(z) is analytic in the close region
D_{R_1,R_2} = \{ 0 < R_1 < |z| < R_2 \} for R_2 < |z| < R_1 and
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{m=1}^{\infty} b_m \frac{1}{z^{m+1}}
\]
also prove that \( f(z) \) has a pole of order m at \( z = a \).
B 1. If \( f(z) \) has a pole at \( z = a \), then prove that \( |f(z)| \to \infty \) as \( z \to a \).
2. If an analytic function \( f(z) \) has a zero of order m at \( z = a \) then prove that
\[
\frac{1}{f(z)}
\]
has a pole of order m at \( z = a \).
C If \( f(z) \) and \( g(z) \) are analytic in \( \Omega \) and if \( f(z) = g(z) \) on a set which has a limit point in \( \Omega \), then prove that \( f(z) \) is identically equal to \( g(z) \).

Q2
A If a function \( f(z) \) is analytic except at finite number of singularities, then prove that
the sum of residues of these singularities is zero.
B Discuss the nature of singularities of the following functions
1. \( f(z) = \frac{1}{z(1-z^2)} \)
2. \( f(z) = \frac{1+z^4}{1-z^4} \)
3. \( f(z) = \frac{\sin z}{(z-\pi)^2} \)
C If an analytic function \( f(z) \) has a pole at \( z=\infty \), then prove that the residue of \( f(z) \) at
infinity is the negative of the coefficient of \( 1/z \) in the expansion of \( f(z) \) of the value
of \( z \) in the neighborhood of \( z=\infty \).
Q3  Attempt any two

A  State and prove Liouville's Theorem.

B  If \( f(z) \) has pole of order \( m \) at \( z = a \), then prove that the residue at \( a \) is the
\[
\lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]
\]

C  State and prove Jorden Leema

Q4  Attempt any two

A  Prove that \( \int_0^{2\pi} \frac{\cos^2 \theta}{1-2p \cos \theta + p^2} \, d\theta = \frac{\pi(1-p^2)}{1-p} \), for \( 0 < p < 1 \) and inside the unit circle

B  Prove that \( \int_0^{2\pi} (1+2 \cos \theta)^n \cos \theta \cdot \frac{\sin \theta}{\sqrt{2}} \, d\theta = \frac{2\pi(3-\sqrt{5})^n}{\sqrt{8}} \), for \( 0 < p < 1 \)

C  Evaluate \( I = \int_0^\infty \frac{\sin x}{a+b \cos x} \, dx \) where \( a > b > 0 \)

Q5  Attempt any two

A  Prove that if \( a > 0 \), then \( \int_0^\infty \frac{1}{x^4 + a^2} \, dx = \frac{\pi\sqrt{2}}{4a^2} \)

B  Using the method of calculus of residue prove that \( \int_0^\infty \frac{\log(1+x^2)}{1+x^2} \, dx = \pi \log 2 \)

C  Prove that \( \int_0^\infty \frac{\cos mx}{x^4 + a^2} \, dx = \frac{2\pi}{2a^2} \, e^{-\frac{ma^2}{4}} \, \sin \left( \frac{ma}{\sqrt{2}} + \frac{\pi}{4} \right) \)