1. (a) Define Euclidean space \( \mathbb{C}^n \). Prove that Euclidean space \( \mathbb{C}^n \) is a metric space. 

(b) Define equivalent norms. Show that 

\[
\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \quad \text{and} \quad \|x\|_2 = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{1/2}
\]

define on a vector space of ordered \( n \)-tuples of numbers are equivalent.

(c) Define algebraically reflexive space. Prove that a finite dimensional vector space is algebraically reflexive.

OR

RR-0306] 1 [Contd....
1  (a) Show that the dual space $X'$ of a normed space $X$ is a Banach space.

(b) Prove that a compact subspace $M$ of a metric space $X$ is closed and bounded.

(c) Let $B(A)$ be the set of all bounded functions defined on a set $A$, define a distance function

$$d(x, y) = \sup_{t \in A} |x(t) - y(t)|; x, y \in B(A).$$

Show that $d$ is a metric on $B(A)$.

2  (a) Show that the space $l^p$ is a complete space.

(b) Prove that an Euclidean space $\mathbb{R}^n$ is a Hilbert space.

(c) Let $T : D(T) \to Y$ be a bounded linear operator, where $D(T) \subset X$, $X$ is a normed space and $Y$ is a Banach space, then prove that $T$ has a bounded linear extension $\overline{T} : D(T) \to Y$ with $\|\overline{T}\| = \|T\|$.

OR

2  (a) Define inner product space. Prove that an inner product space is a normed space.

(b) Define Direct sum. Let $Y$ be any closed subspace of a Hilbert space $H$, then prove that $H = Y \oplus Z$, where $Z = Y^\perp$.  

[Contd....]
(c) A dot product defines a functional \( f : \mathbb{R}^3 \to \mathbb{R} \) by
\[
f(x) = x_1 a + x_2 a + x_3 a; \quad \text{where} \quad x \in \mathbb{R}^3, \ a \in \mathbb{R},
\]
then show that \( f \) is linear, bounded with \( \|f\| = \|a\| \).

3 (a) Let \( \| \cdot \| \) be a norm generated by inner product on \( X \), then show that for complex scalar:

(i) \( R_c \langle x, y \rangle = \frac{1}{4} \left[ \| x + y \|^2 - \| x - y \|^2 \right] \)

(ii) \( I_m \langle x, y \rangle = \frac{1}{4} \left[ \| x + iy \|^2 - \| x - iy \|^2 \right] \)

(b) In an inner product space \( X \), prove that
\[
x \perp y \Rightarrow \| x + y \|^2 = \| x \|^2 + \| y \|^2.
\]

(c) Let \( x_1(t) = t^2, x_2(t) = t \) and \( x_3(t) = 1 \) Orthonormalize \( x_1, x_2 \) and \( x_3 \) in this order on the interval \([-1,1]\) with respect to the inner product \( \langle x, y \rangle = \int_{-1}^{1} x(t) y(t) dt \).

OR

3 (a) Define self adjoint operator and prove the following.

Let \( U : H \to H \) and \( V : H \to H \) be unitary operator on a Hilbert source \( H \); then :

(i) \( U \) is isometric and \( \| U_x \| = \| x \| \)

(ii) \( \| U \| = 1 \) where \( H \neq \{0\} \)

(iii) \( U^{-1} \) is unitary.
(b) Let $X$ be a normed linear space and $x_0 \in X$ be arbitrary, where $x_0 \neq 0$, then prove that $\exists$ a bounded linear functional $\hat{f}$ on $X$ such that $\|\hat{f}\|=1$ and $\hat{f}(x_0) = \|x_0\|.$

(c) Let $H$ and $\tilde{H}$ be two Hilbert space and $S : H \to \tilde{H}$ and $T : H \to \tilde{H}$ be bounded linear operator and $\alpha$ be any scalar, then prove that

(i) $\left\langle T^*_y, x \right\rangle = \left\langle y, T_x \right\rangle$

(ii) $(S + T)^* = S^* + T^*$

(iii) $(\alpha T)^* = \bar{\alpha} T^*$.

4 (a) State and prove Riesz theorem. 20

(b) Show that Relation between adjoint operator and Hilbert adjoint operator.

(c) Let $X$ and $Y$ be inner product space and $Q : X \to Y$ be a bounded linear operator, then prove that

(i) $Q = 0$ iff $\left\langle Q_x, y \right\rangle = 0, \ \forall x \in X, \forall y \in Y$

(ii) $Q : X \to X$, where $X$ is complex and $\left\langle Q_x, x \right\rangle = 0$ then $Q = 0$.

OR

4 (a) Let $H$ be a Hilbert space and if $H$ is separable then prove that every orthonormal set in $H$ is countable. 20
(b) For every $x$ in a normed space $X$,

\[ \|x\| = \sup_{\substack{f \in X', \; f \neq 0 \atop f \neq 0}} \frac{|f(x)|}{\|f\|} \]

and if $x_0$ is

Such that $f(x_0) = 0 \quad \forall \; f \in X'$, then prove that $x_0 = 0$.

(5) State and prove the Hahn Banach Theorem on

Normed spaces.

(b) Define Partially ordered set. Prove that "Every vector space $X \neq \{0\}$ has Hamel basis.

(c) Let $T : D(T) \to Y$ be a bounded linear operator with domain $D(T) \subset X$, where $X$ and $Y$ are normed space, then prove that

(i) If $D(T)$ is a closed subset of $X$, then $T$ is closed.

(ii) If $T$ is closed and $Y$ is complete then $D(T)$ is a closed subset of $X$.

OR

(a) Define bounded linear operator. Let $T : H_1 \to H_1$ be a bounded linear operator on Hilbert space $H_1$, then prove that

(i) If $T$ is self adjoint then $\langle T_x, x \rangle$ is real.

(ii) If $H$ is complex and $\langle T_x, x \rangle$ is real $\forall \; x \in H$, then $T : H_1 \to H_1$ is self adjoint.
(b) Let \((x_n)\) be a weakly convergent sequence in a normed space \(X\), say \(x_n \xrightarrow{w} x\), then prove that:

(i) then weak limit \(x\) of \((x_n)\) is unique.

(ii) every subsequence of \((x_n)\) converges weakly to \(x\).

(c) Let \((T_n)\) be a sequence of bounded linear operator \(T_n : X \to Y\) from a Banach space \(X\) into a normed space \(Y\), such that \(\|T_n x\|\) is bounded for every \(x \in X\), say \(\|T_n x\| \leq C_x\) where \(n = 1, 2, \ldots\) and \(C_x\) is real number, then prove that the sequence of the norms \(\|T_n\|\) is bounded.